

# On Carlotto-Schoen-type scalar-curvature gluings\*

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## Abstract

We carry out a Carlotto-Schoen-type gluing with interpolating scalar curvature on cone-like sets, or deformations thereof, in the category of smooth Riemannian asymptotically Euclidean metrics.

## 1 Introduction

In an outstanding paper [5] Carlotto and Schoen have constructed non-trivial asymptotically Euclidean scalar-flat metrics which are Minkowskian outside of a solid cone. In this note we show how to generalise the construction there in several directions:

1. Rather than gluing an asymptotically Euclidean metric to a flat one, any two asymptotically Euclidean metrics  $g_1$  and  $g_2$  are glued together.
2. In the spirit of [11], the gluings at zero-scalar curvature are replaced by gluings where the scalar curvature of the final metric equals

$$\chi R(g_1) + (1 - \chi)R(g_2)$$

with a cut-off function  $\chi$ .

3. We provide a detailed proof with exponential weights near the boundary.
4. The geometry of the gluing region is allowed to be more general than the interface between two cones.

A special case of our main Theorem 3.1 below provides the following variant of the Riemannian-geometry version of the main theorem of [5]:

**THEOREM 1.1** *Let  $n \geq 3$ . Given two smooth  $n$ -dimensional asymptotically Euclidean Riemannian metrics  $g_1$  and  $g_2$  there exist cones and smooth asymptotically Euclidean metrics  $g$  which coincide with  $g_1$  outside of the cones and with  $g_2$  inside slightly smaller cones, see Figure 1.1, with the scalar curvature  $R(g)$  between  $R(g_1)$  and  $R(g_2)$  in the intermediate region.*

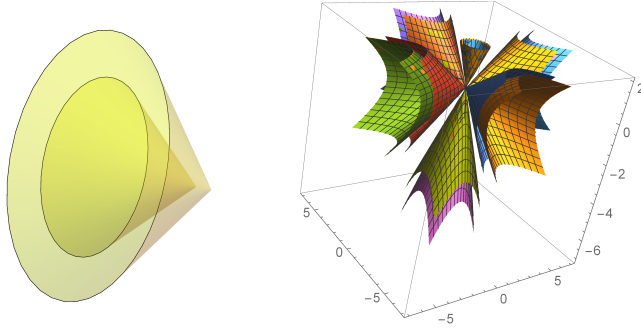


Figure 1.1: If the second metric is flat and both metrics are scalar-flat, then the new metric is flat inside the smaller cone, scalar-flat everywhere, and coincides with the first one outside the larger cone. Both cones extend to infinity, and their tips are located very far in the asymptotically Euclidean region. Right picture: Iterating the construction, one can embed any finite number of distinct initial data sets into Minkowskian data, or paste-in Minkowskian data inside several cones into a given data set.

In particular if  $R(g_1) = R(g_2) = 0$ , then  $R(g) = 0$ .

In Section 6 we show how Theorem 1.1 follows from our main Theorem 3.1 below. The key to the proof of this last theorem is a weighted Poincaré inequality involving radially-scaled exponential weights, proved in Proposition 5.6 below. The rest of the proof is a verification of the hypotheses of the rather general results proved in [7].

As already mentioned, Theorem 3.1 below allows more general cones than the ones in [5], as described at the beginning of Section 3. In fact, our arguments apply to a large class of deformed cones as well, e.g. “logarithmically-rotated ones”, cf. Theorem 4.1 and the discussion in Section 4.

## 2 Definitions, notations and conventions

We use the summation convention throughout, indices are lowered with  $g_{ij}$  and raised with its inverse  $g^{ij}$ .

We will have to frequently control the asymptotic behavior of the objects at hand. Given a tensor field  $T$  and a function  $f$ , we will write

$$T = O(f),$$

when there exists a constant  $C$  such that the  $g$ -norm of  $T$  is dominated by  $Cf$ .

A metric  $g$  on a manifold  $M$  will be said to be asymptotically Euclidean (AE) if  $M$  contains a region, diffeomorphic to the complement of a ball in  $\mathbb{R}^n$ , on which the metric  $g$  approaches the Euclidean metric  $\delta$  as one recedes to infinity.

Let  $\phi$  and  $\psi$  be two smooth strictly positive functions on an  $n$ -dimensional manifold  $M$ . The function  $\psi$  will be used to control the growth of the fields involved near boundaries or in the asymptotic regions, while  $\phi$  will control how the growth is affected by derivations. For  $k \in \mathbb{N}$  let  $H_{\phi, \psi}^k(g)$  be the space of

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$H_{\text{loc}}^k$  functions or tensor fields such that the norm<sup>1</sup>

$$\|u\|_{H_{\phi,\psi}^k(g)} := \left( \int_M \left( \sum_{i=0}^k \phi^{2i} |\nabla^{(i)} u|_g^2 \right) \psi^2 d\mu_g \right)^{\frac{1}{2}} \quad (2.1)$$

is finite, where  $\nabla^{(i)}$  stands for the tensor  $\underbrace{\nabla \dots \nabla}_{i \text{ times}} u$ , with  $\nabla$  — the Levi-Civita covariant derivative of  $g$ . We assume throughout that the metric is at least  $W_{\text{loc}}^{1,\infty}$ ; higher differentiability will be usually indicated whenever needed.

For  $k \in \mathbb{N}$  we denote by  $\mathring{H}_{\phi,\psi}^k$  the closure in  $H_{\phi,\psi}^k$  of the space of  $H^k$  functions or tensors which are compactly (up to a negligible set) supported in  $M$ , with the norm induced from  $H_{\phi,\psi}^k$ . The  $\mathring{H}_{\phi,\psi}^k$ 's are Hilbert spaces with the obvious scalar product associated with the norm (2.1). We will also use the following notation

$$\mathring{H}^k := \mathring{H}_{1,1}^k, \quad L_\psi^2 := \mathring{H}_{1,\psi}^0 = H_{1,\psi}^0,$$

so that  $L^2 \equiv \mathring{H}^0 := \mathring{H}_{1,1}^0$ .

For  $\phi$  and  $\varphi$  — smooth strictly positive functions on  $M$ , and for  $k \in \mathbb{N}$  and  $\alpha \in [0, 1]$ , we define  $C_{\phi,\varphi}^k$  to be the space of  $C^k$  functions or tensor fields for which the norm

$$\|u\|_{C_{\phi,\varphi}^k(g)} = \sup_{x \in M} \sum_{i=0}^k \|\varphi \phi^i \nabla^{(i)} u(x)\|_g$$

is finite.

Let  $g, g_0$  be two Riemannian metrics and let  $W$  be a space of two-covariant symmetric tensors. We will write  $g \in M_{g_0+W}$  if  $g - g_0 \in W$ . An important example is provided by metrics  $g \in M_{\delta+C_{r,r,\epsilon}^{k+4}}$ , where  $\delta$  denotes the Euclidean metric. Such metrics are asymptotically Euclidean, with the difference  $g - \delta$  decaying as  $O(r^{-\epsilon})$ , and with derivatives of order  $1 \leq \ell \leq k+4$  decaying as  $O(r^{-\epsilon-\ell})$ .

### 3 Gluing on sets which are scale-invariant at large distances

Let  $S(p, R) \subset \mathbb{R}^n$  denote a sphere of radius  $R$  centred at  $p$ .

Let  $\Omega \subset \mathbb{R}^n$  be a domain with smooth boundary (thus  $\Omega$  is open and connected, and  $\partial\Omega = \bar{\Omega} \setminus \Omega$  is a smooth manifold). We further assume that  $\partial\Omega$  has *exactly two* connected components, and that there exists  $R_0 \geq 1$  such that

$$\Omega_S := \Omega \cap S(0, R_0) \quad (3.1)$$

also has again exactly two connected components, with

$$\Omega \setminus B(0, R_0) = \{\lambda p \mid p \in \Omega_S, \lambda \geq 1\}. \quad (3.2)$$

When (3.2) holds, we say that  $\Omega$  is *invariant under scaling at large distances*.

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<sup>1</sup>The reader is referred to [1, 2, 12] for a discussion of Sobolev spaces on Riemannian manifolds.

We let  $x : \overline{\Omega} \rightarrow \mathbb{R}$  be any smooth defining function for  $\partial\Omega$  which has been chosen so that

$$x(\lambda p) = \lambda x(p) \text{ for } \lambda \geq 1 \text{ and for } p, \lambda p \in \Omega \setminus B(0, R_0). \quad (3.3)$$

Equivalently, for  $p \in \Omega_S$  and  $\lambda$  larger than one, we require  $x(\lambda p) = \lambda x_S(p)$ , where  $x_S$  is a defining function for  $\partial\Omega_S$  within  $S(0, R_0)$ .

By definition of  $\Omega$  there exists a constant  $c$  such that the distance function  $d(p)$  from a point  $p \in \Omega$  to  $\partial\Omega$  is smooth for all  $d(p) \leq cr(p)$ . The function  $x$  can be chosen to be equal to  $d$  in that region.

A simple example, considered in [5], satisfying the above is the following: let  $0 < \theta_1 < \theta_2 < \pi$  and let  $\Omega_{\theta_1, \theta_2}$  any domain with smooth boundary such that

$$\Omega_{\theta_1, \theta_2} \setminus B(0, R_0) = (\mathring{C}_{\theta_2} \setminus C_{\theta_1}) \setminus B(0, R_0), \quad (3.4)$$

where  $C_\theta$  is a closed solid cone with aperture  $\theta > 0$ ; compare the left Figure 1.1. Here, and elsewhere,  $\mathring{U}$  denotes the interior of a set  $U$ . In this example  $\Omega_S := \Omega_{\theta_1, \theta_2} \cap S(0, R_0)$  is connected, with a boundary which has two connected components.

Some other examples can be found in Figure 3.1

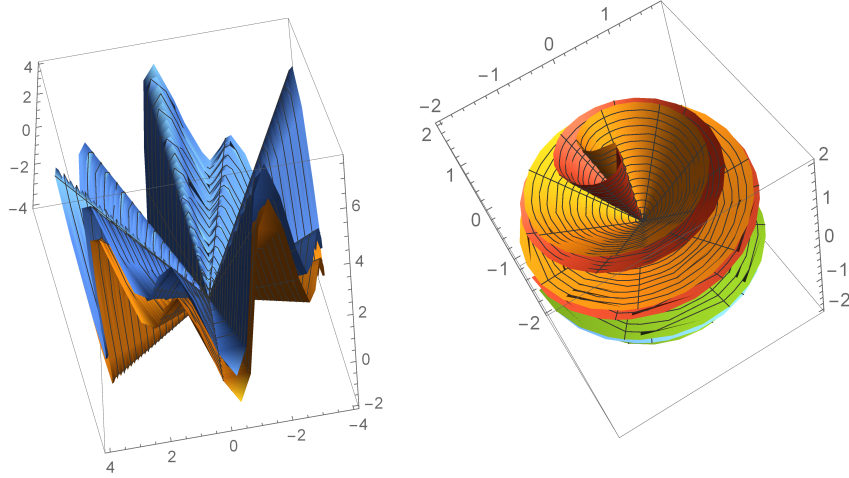


Figure 3.1: Left figure: A possible set  $\Omega$ , located between the two surfaces. The metric coincides with  $g_1$  above the first surface, with  $g_2$  below the second one, and is scalar-flat everywhere if both  $g_1$  and  $g_2$  were. Right figure: A scale-invariant thickening of the displayed hypersurface provides a set  $\Omega$  so that  $\mathbb{R}^3 \setminus \Omega$  has two components. The metric will coincide with  $g_1$  in one component, and with  $g_2$  in the other.

We will denote by  $r$  a smooth positive function which coincides with  $|\vec{x}|$  for  $|\vec{x}| \geq 1$ .

For  $\beta, s, \mu \in \mathbb{R}$  we define

$$\phi = \left(\frac{x}{r}\right)^2 r = \frac{x^2}{r}, \quad \psi = r^{-n/2-\beta} \left(\frac{x}{r}\right)^\sigma e^{-sr/x} =: r^\mu x^\sigma e^{-sr/x} \quad (3.5)$$

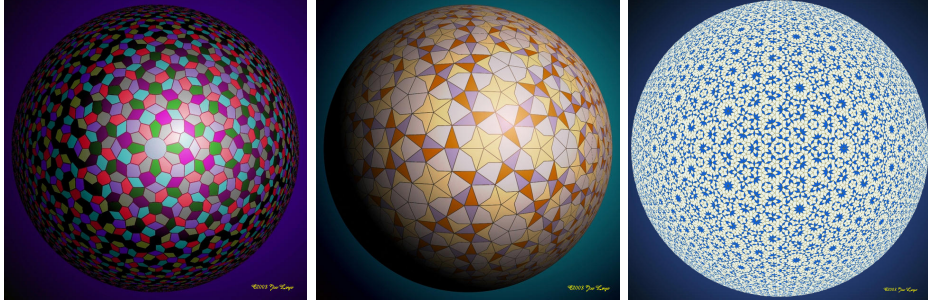


Figure 3.2: After iterating the gluing a finite number of times (first two pictures), or an infinite number of times (last picture) one can obtain aesthetically pleasing sets  $\Omega_S$ . The infinite iteration might require going to larger and larger distances for each individual gluing. Images by Jos Leys, with kind permission of the author.

on  $\Omega$ . The factor  $r^{-\beta}$  is directly related to the large-distance radial behavior of the solutions, while the exponential factor with  $s > 0$  will force the solution to vanish at all orders at  $\partial\Omega$  (compare (3.12) below).

We will make a gluing-by-interpolation of scalar curvature. The main interest is that of scalar-flat metrics, which then remain scalar-flat, or for metrics with positive scalar curvature, which then remains positive. Since the current problem is related to the construction of initial data sets for Einstein equations, in some general-relativistic matter models, such as Vlasov or dust, an interpolation of scalar curvature is of direct interest.

In order to carry out the interpolation, recall that  $\partial\Omega$  has exactly two connected components. We denote by  $\chi$  a smooth function with the following properties:

1.  $0 \leq \chi \leq 1$ ;
2.  $\chi$  equals one in a neighborhood of one of the components and equals zero in a neighborhood of the other component;
3. on  $\Omega \setminus B(0, R_0)$  the function  $\chi$  is required to depend only upon  $x/r$ .

Starting with two AE metrics  $g$  and  $\hat{g}$  we define

$$g_\chi := \chi\hat{g} + (1 - \chi)g, \quad (3.6)$$

$$R_\chi := \chi R(\hat{g}) + (1 - \chi)R(g), \quad \delta R_\chi := R_\chi - R(g_\chi). \quad (3.7)$$

We denote by  $P_g$  the linearisation of the scalar-curvature operator at a metric  $g$ , and will write  $P$  for  $P_g$  when  $g$  is obvious in the context. Its formal adjoint  $P_g^*$  reads

$$P_g^*(f) = \nabla\nabla f - \Delta f g - f \text{Ric}(g), \quad (3.8)$$

where  $\Delta = \nabla^i \nabla_i$ .

Letting  $\Omega$ ,  $x$  and  $r$  be as just described, with  $\psi$ ,  $\phi$  given by (3.5), we have:

THEOREM 3.1 *Let  $\epsilon > 0$ ,  $k > n/2$ ,  $\beta \in [-(n-2), 0)$ , and  $\tilde{\beta} < \min(\beta, -\epsilon)$ . Suppose that  $g \in M_{\delta+C_{r,r^\epsilon}^{k+4}} \cap C^\infty$ . For all real numbers  $\sigma$  and  $s > 0$  and*

*all smooth metrics  $\hat{g}$  close enough to  $g$  in  $C_{r,r^{-\tilde{\beta}}}^{k+4}(\Omega)$*

*there exists on  $\Omega$  a unique smooth two-covariant symmetric tensor field  $h$  of the form*

$$h = \phi^4 \psi^2 P_{g_\chi}^*(\delta N) \in \psi^2 \phi^2 \mathring{H}_{\phi,\psi}^{k+2}(g) \quad (3.9)$$

*such that the metric  $g_\chi + h$  solves*

$$R[g_\chi + h] = \chi R(\hat{g}) + (1 - \chi)R(g). \quad (3.10)$$

*The tensor field  $h$  vanishes at  $\partial\Omega$  and can be  $C^\infty$ -extended by zero across  $\partial\Omega$ , leading to a smooth asymptotically Euclidean metric  $g_\chi + h$ .*

REMARK 3.2 Some comments about the asymptotic behaviour of the metrics involved are in order. The requirement  $g \in M_{\delta+C_{r,r^\epsilon}^{k+4}} \cap C^\infty$  (equivalently,  $g - \delta \in C_{r,r^\epsilon}^{k+4} \cap C^\infty$ ) guarantees that  $g$  is asymptotically Euclidean, with  $g$  approaching the Euclidean metric as  $O(r^{-\epsilon})$ . Note that  $\epsilon$  can be arbitrary small. Next,  $g - \hat{g}$  is required to fall off as  $r^{\tilde{\beta}}$ , so that  $g_\chi$  will approach  $\delta$  as  $O(r^{\min(-\epsilon, \tilde{\beta})})$ . The final metric  $g_\chi + h$  will a priori approach  $\delta$  somewhat slower. Indeed, the proof below shows that there exists a constant  $C$  such that

$$\|h\|_{\psi^2 \phi^2 \mathring{H}_{\phi,\psi}^{k+2}(g_\chi)} \leq C \|\delta R_\chi\|_{\psi^2 \mathring{H}_{\phi,\psi}^k(g_\chi)}. \quad (3.11)$$

(The right-hand side is finite and small only if the weight  $-\beta$  is larger than  $\max(-\tilde{\beta}, \epsilon)$ , which explains the hypothesis.)

Equation (3.11) implies (compare (3.24) below)

$$h = o(r^{-n-\beta+2}(x/r)^{\sigma+4-n}e^{-sr/x}) = o(r^{-n-\beta+2}). \quad (3.12)$$

Since  $\beta < 0$ , (3.12) shows that the decay of the metric to the flat-one is *most likely* slower than the Schwarzschildian decay rate  $O(r^{-(n-2)})$ .

REMARK 3.3 A well defined ADM energy-momentum requires the metric to approach the flat-one as  $o(r^{-(n-2)/2})$ . Comparing with (3.12), this will be satisfied if in Theorem 3.1 we require instead

$$-\frac{n-2}{2} \leq \beta \leq 0. \quad (3.13)$$

REMARK 3.4 An analogous result holds if  $g$  is not necessarily smooth: If  $g \in M_{\delta+C_{r,r^\epsilon}^{k+4}}$ , then the construction can be modified as in [8] so that the final metric  $g_\chi + h$  will be of  $C^{k+4}$  differentiability class.

PROOF: We start by showing that Theorem 3.6 of [7] with  $K \equiv 0 \equiv Y$  applies, with  $g_0$  taken to be the Euclidean metric. This requires, first, checking the “scaling property” of the functions  $\phi$  and  $\psi$ . This is a routine exercise,

using scaling in  $r$  and [7, Appendix B]. Next, note that with a flat metric  $g_0$  Equation (3.8) reads

$$P_{g_0}^*(f) = \nabla \nabla f - \Delta f g_0. \quad (3.14)$$

Since  $\beta \neq 0$ , the property [7, Equation (3.4)] needed in Theorem 3.6 follows directly from Proposition 5.6 below used twice, first for  $u$  and then for  $\nabla u$ . Remark that the integrals involving  $u$  in the right-hand side of (5.22) will vanish if the support of  $u$  lies in the region  $|\vec{x}| \geq R$ .

Now, the operator  $\pi_{\mathcal{K}_0^\perp}$  appearing in the statement of [7, Theorem 3.6] projects on the space  $\mathcal{K}_0^\perp$  of *static KIDs*, namely the kernel of  $P^*$ . We claim that under our hypotheses this space is trivial, and so the operator  $L_{\phi,\psi}$  of [7, Theorem 3.6] is an isomorphism:

Indeed, it is well-known that static KIDs that are  $o(1)$  along some cone  $C_\varepsilon$  have to vanish everywhere. (This is essentially [10, Proposition 2.1], together with unique continuation for solutions of the KID equation.) So it suffices to show that functions  $\delta N \in \dot{H}_{\phi,\psi}^{k+4}(g)$  are  $o(1)$  in some cone contained in  $\Omega$ . For this let us choose a cone  $C_\varepsilon \subset \Omega$  on which  $x/r$  is bounded away from zero. Then the restriction  $\delta N|_{C_\varepsilon}$  is in the space

$$\mathcal{H}_k^\beta(C_\varepsilon) := \dot{H}_{r,r-n/2-\beta}^k(C_\varepsilon). \quad (3.15)$$

We note the inclusions [3]

$$C_k^{\beta'}(C_\varepsilon) \subset \mathcal{H}_k^\beta(C_\varepsilon), \quad \beta' < \beta, \quad \text{and} \quad \mathcal{H}_k^\beta(C_\varepsilon) \subset C_{[k-n/2]}^\beta(C_\varepsilon), \quad k > n/2. \quad (3.16)$$

In fact [3]

$$f \in \mathcal{H}_k^\beta(C_\varepsilon), \quad k > n/2 \quad \implies \quad f = o(r^\beta). \quad (3.17)$$

So, the requirement that there are no KIDs in the space under consideration will be satisfied for

$$\beta \leq 0. \quad (3.18)$$

(A more careful inspection shows (compare (3.24) below) that

$$\delta N \in \dot{H}_{\phi,\psi}^k, \quad k > n/2 \quad \implies \quad \delta N = o\left(r^\beta \left(\frac{x}{r}\right)^{-\sigma-n} e^{sr/x}\right), \quad (3.19)$$

but a possible blow-up of  $\delta N$  when  $x$  tends to zero is irrelevant as long as  $\delta N$  is forced to go to zero when receding to infinity away from  $\partial\Omega$ .)

Summarising, [7, Theorem 3.6] applies and shows that the map  $L_{\phi,\psi}$  there is an isomorphism.

We now want to use the inverse function theorem as in [7, Theorem 3.7] to solve the equation (3.10); equivalently

$$R[g_\chi + h] - R(g_\chi) = \delta R_\chi. \quad (3.20)$$

This requires checking differentiability of the map defined there. This will follow by standard arguments if the metrics of the form  $g_\chi + h$ , with  $h$  given by (3.9), are asymptotically Euclidean. Now, given a perturbation  $\delta R$  of the Ricci scalar

on  $\Omega$ , the linearized perturbed metric  $h$  is obtained from the solution  $\delta N$  of the equation

$$\psi^2 L_{\phi, \psi} \delta N \equiv P \underbrace{\phi^4 \psi^2 P^* \delta N}_{=: h} = \delta R \in \psi^2 \mathring{H}_{\phi, \psi}^k(g). \quad (3.21)$$

In cones  $C_\varepsilon$  staying away from the boundary we have  $\delta N \in \mathring{\mathcal{H}}_{k+4}^\beta(C_\varepsilon)$ , and since on  $C_\varepsilon$  we have  $\psi \sim r^{-n/2-\beta}$  and  $\phi \sim r$  we obtain

$$h = \phi^4 \psi^2 P^* \delta N = r^{-n-2\beta+4} o(r^{\beta-2}) = o(r^{-\beta-n+2}). \quad (3.22)$$

We conclude that  $h$  will decay to zero in such cones provided that

$$-(n-2) \leq \beta. \quad (3.23)$$

A more careful treatment, without invoking interior cones, uses the weighted Sobolev embedding

$$\begin{aligned} h \in \psi^2 \phi^2 \mathring{H}_{\phi, \psi}^{k+2}(g_\chi) &\subset r^{-n-2\beta+2} (x/r)^{2\sigma+4} e^{-2sr/x} C_{x^2/r, r^{-\beta}(x/r)^{\sigma+n} e^{-sr/x}}^{k+2-\lfloor n/2 \rfloor + \alpha} \\ &= C_{r^{n+\beta-2}(x/r)^{-\sigma-4+n} e^{sr/x}}^{k+2-\lfloor n/2 \rfloor + \alpha} \end{aligned} \quad (3.24)$$

where  $\alpha$  is any number in  $(0, 1)$  when  $n$  is even, and  $\alpha = 1/2$  when  $n$  is odd. (The inclusion above is obtained by a calculation as in [7, Equation (B.4)] where, using the notation there, in the fifth line the elliptic regularity estimate is replaced by the Sobolev inequality on  $B(0, 1/4)$  for the function  $u \circ \varphi_p$ .) This guarantees differentiability of the constraints map, as required for applicability of the inverse function theorem.

Smoothness of solutions is standard: away from  $\partial\Omega$  this follows from elliptic regularity, while near  $\partial\Omega$  this is guaranteed by the exponential decay of  $h$  and its derivatives at  $\partial\Omega$ . The proof is now complete.  $\square$

## 4 Beyond scale-invariant sets

Let  $(\mathcal{S}, g)$  be an asymptotically Euclidean manifold and let  $\Omega \subset \mathcal{S}$ ,  $r$  and  $x$  be as described at the beginning of Section 3. Let  $g_0$  be any smooth metric on  $\mathcal{S}$  which coincides with the Euclidean metric at all large distances in the asymptotically Euclidean end. We will use the metric  $g_0$  in the definitions of the functional spaces.

Suppose that  $\Psi : \mathcal{S} \rightarrow \mathcal{S}$  is a smooth diffeomorphism such that

$$\Psi \in C_{r, r^{-1}}^{k+5}, \text{ with } \Psi^* g \text{ uniformly equivalent to } g. \quad (4.1)$$

The symbol  $\chi$  will denote a cut-off function which equals the function  $\chi$  defined shortly before (3.6) *composed with*  $\Psi$ .

We have:

**THEOREM 4.1** *Under (4.1) and the remaining hypotheses of Theorem 3.1, the conclusions of Theorem 3.1, as well as Remarks 3.2-3.4, hold with  $\Omega$  replaced by  $\Psi(\Omega)$ .*



REMARK 4.2 As in our remaining analysis, the requirement of closedness of  $g$  and  $\hat{g}$  can be achieved by translating  $\Omega$  sufficiently far to the asymptotic region, or by scaling as in Section 6).

PROOF: We note that there are no KIDs on  $\Psi(\Omega)$  which tend to zero when staying away from the boundary of  $\Psi(\Omega)$ . This can be proved by calculations similar to those of [4, Proposition 2.2], where integration over the rays

$$\gamma_p = \{rp, r \in [1, \infty)\}, p \in S(R_0),$$

is replaced by integration over  $\Psi(\gamma_p)$ .

The result follows now by applying the following result to the metrics  $\Psi^*g$  and  $\Psi^*\hat{g}$  on  $\Omega$ :  $\square$

THEOREM 4.3 *Let  $\epsilon > 0$ ,  $k > n/2$ ,  $\beta \in [-(n-2), 0)$ , and  $\tilde{\beta} < \min(\beta, -\epsilon)$ . Let  $g_0$  be any smooth metric on  $\mathcal{S}$  which coincides with the Euclidean metric in the asymptotic region. Suppose that  $g \in C_{r,1}^{k+4} \cap C^\infty$  is uniformly equivalent to  $g_0$  and has no static KIDs on  $\Omega$ . For all real numbers  $\sigma$  and  $s > 0$  and*

$$\text{all smooth metrics } \hat{g} \text{ close enough to } g \text{ in } C_{r,r^{-\tilde{\beta}}}^{k+4}(\Omega)$$

*there exists on  $\Omega$  a unique smooth two-covariant symmetric tensor field  $h$  of the form*

$$h = \phi^4 \psi^2 P_{g_\chi}^*(\delta N) \in \psi^2 \phi^2 \mathring{H}_{\phi,\psi}^{k+2}(g) \quad (4.2)$$

*such that the metric  $g_\chi + h$  solves*

$$R[g_\chi + h] = \chi R(\hat{g}) + (1 - \chi)R(g). \quad (4.3)$$

*The tensor field  $h$  vanishes at  $\partial\Omega$  and can be  $C^\infty$ -extended by zero across  $\partial\Omega$ , leading to a smooth metric  $g_\chi + h$  which approaches  $g$  as  $r$  recedes to infinity in the asymptotic region.*

PROOF: Theorem 4.3 follows from the inverse function theorem applied to the map

$$r_\chi : \mathring{H}_{\phi,\psi}^{k+4}(g) \ni \delta N \mapsto \psi^{-2}[(R(g_\chi + \phi^4 \psi^2 P_{g_\chi}^*(\delta N)) - R(g_\chi))] \in \mathring{H}_{\phi,\psi}^k(g), \quad (4.4)$$

exactly as in the proof of Theorem 3.1. Recall that the map  $r_\chi$  will have  $\psi^{-2}\delta R_\chi$  in its range when  $g_\chi$  is close to  $g$  in  $C_{\phi,1}^{k+4}$  because the inverse,  $Dr_\chi(0)^{-1}$ , of the linearised map  $Dr_\chi(0)$  has a bound uniform over a neighborhood of  $g_\chi$  in this topology, as follows in part from the uniformity of the constant in the weighted Poincaré inequality of Section 5.  $\square$

REMARK 4.4 An alternative proof of Theorem 4.1 proceeds by checking that Propositions 5.4 and 5.6 below hold with  $\Omega$  replaced by  $\Psi(\Omega)$ ,  $r$  replaced by  $r_\Psi := r \circ \Psi^{-1}$ , and  $x$  replaced by  $x_\Psi := x \circ \Psi^{-1}$ . Since we have already seen (see the proof of Theorem 4.1) that there are no KIDs vanishing at infinity on  $\Psi(\Omega)$ , the proof of Theorem 3.1 applies verbatim.

To verify the propositions, one starts by noting that there exists a constant  $C_1$  such that we have

$$C_1^{-1}r \leq r_\Psi \leq C_1 r. \quad (4.5)$$

This can be established by standard considerations (cf., e.g., the proof of [6, Equation (30)]). Hence, decay rates in  $r_\Psi$  are equivalent to decay rates in  $r$ .

Let  $u_\Psi := u \circ \Psi$ , and let us denote by  $\mu_g$  the Riemannian measure associated with a metric  $g$ . We then have for functions  $u$  compactly supported in  $\Psi(\Omega)$ , using change-of-variables, Proposition 5.6, and a trace theorem,

$$\begin{aligned} \int_{\Psi(\Omega)} x_\Psi^{2\sigma} r_\Psi^{2\mu} e^{-sr_\Psi/x_\Psi} |u|^2 d\mu_g &= \int_{\Omega} x^{2\sigma} r^{2\mu} e^{-sr/x} |u_\Psi|^2 d\mu_{\Psi^*g} \\ &\leq C \int_{\Omega} x^{2\sigma} r^{2\mu} e^{-sr/x} |u_\Psi|^2 d\mu_g \\ &\leq C^2 \left( \int_{\{x \geq x_0, r=R\}} |u_\Psi|^2 + \int_{\{x \geq x_0, r \leq R\}} |u_\Psi|^2 d\mu_g \right. \\ &\quad \left. + \int_{\Omega} x^{2\sigma+4} r^{2\mu-2} e^{-sr/x} |\nabla u_\Psi|_g^2 d\mu_g \right) \\ &\leq C^3 \left( \int_{\{x \geq x_0, r \leq R\}} |u_\Psi|^2 d\mu_{\Psi^*g} + \int_{\Omega} x^{2\sigma+4} r^{2\mu-2} e^{-sr/x} |\nabla u_\Psi|_{\Psi^*g}^2 d\mu_{\Psi^*g} \right) \\ &= C^3 \left( \int_{\Psi(\{x \geq x_0, r \leq R\})} |u|^2 d\mu_g + \int_{\Psi(\Omega)} x_\Psi^{2\sigma+4} r_\Psi^{2\mu-2} e^{-sr_\Psi/x_\Psi} |\nabla u|_g^2 d\mu_g \right), \end{aligned} \quad (4.6)$$

as claimed, assuming of course  $s \neq 0$  and  $\sigma + \mu + n/2 \neq 0$ .

The result for tensor fields  $u$  easily follows (cf., e.g., [9, Remark 4.7]).

There is likewise an equivalent of (5.9) in the current setting.  $\square$

A trivial example of maps  $\Psi$  which satisfy the above is linear maps of  $\mathbb{R}^n$ . This, however, does not lead to a new family of sets  $\Omega$  as compared to Theorem 3.1. Another example is provided by maps which are asymptotic to linear maps, which again does not lead to any additional essential changes at large distances.

A non-trivial example is provided by *logarithmic rotations*, whose action on cones or tennis-ball curves can be seen in Figure 4.1, and which we define as follows: Let  $R(t)$  denote a rotation by angle  $t$  around the  $z$ -axis. For  $\alpha \in \mathbb{R}$  consider the smooth map  $\Psi_\alpha$  defined as

$$\mathbb{R}^3 \setminus \overline{B(1)} \ni \vec{x} \mapsto R(\alpha \ln |\vec{x}|) \vec{x} \in \mathbb{R}^3 \setminus \overline{B(1)}.$$

In spherical coordinates on  $\mathbb{R}^3$  the map  $\Psi_\alpha$  takes the form

$$(r, \theta, \varphi) \mapsto (r, \theta, \varphi + \alpha \ln r).$$

This leads to the following form of the pull-back  $\Psi_\alpha^* \delta$  of the Euclidean metric  $\delta$ :

$$\begin{aligned} \Psi_\alpha^* \delta &= dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \left( d\varphi + \frac{\alpha dr}{r} \right)^2 \right) \\ &= (1 + \alpha^2 \sin^2 \theta) dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) + 2\alpha r dr d\varphi. \end{aligned} \quad (4.7)$$

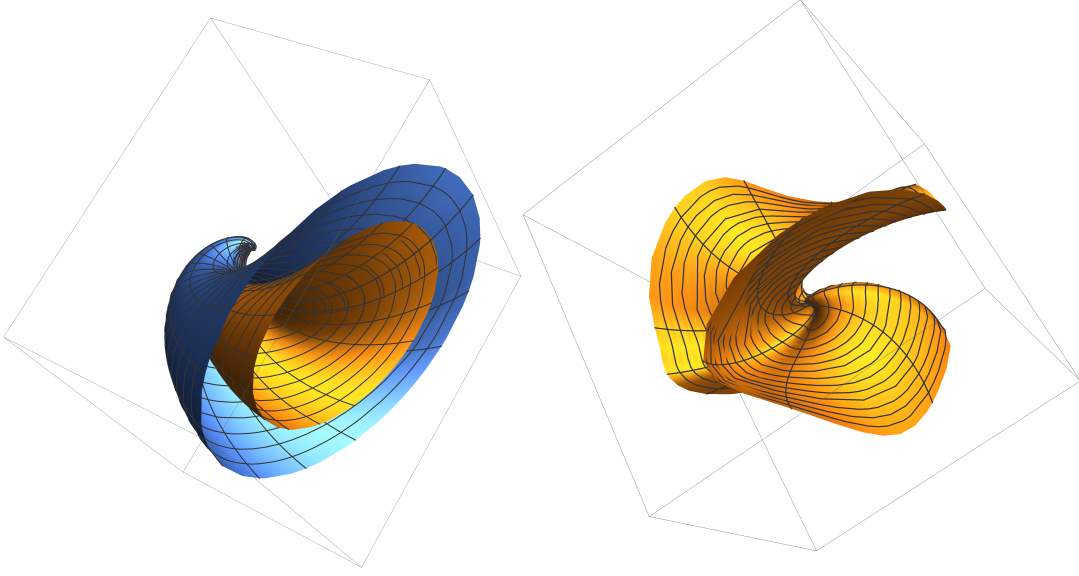


Figure 4.1: Left figure: A logarithmic rotation applied to two cones;  $\Omega$  is the region between the cones. Right figure: A logarithmic rotation applied to the “tennis-ball curve” (namely, the curve along which the halves of a tennis-ball are joined together). A scale-invariant thickening of the surface displayed provides the set  $\Omega$ .

Using

$$2rdrd\varphi \leq dr^2 + r^2d\varphi^2 \leq \delta$$

we find for  $|\alpha| < 1$

$$(1 - |\alpha|)\delta \leq \Psi_\alpha^* \delta \leq (1 + |\alpha| + \alpha^2)\delta \leq 3\delta, \quad (4.8)$$

as desired.

We note that while the image of a scale-invariant set under a logarithmic rotation is not scale-invariant, it is invariant under a discrete scaling, which would have been enough for a direct analysis in Section 5 anyway.

Composing the above rotations along different axes leads to further non-trivial examples on  $\mathbb{R}^3$ .

The above generalises in an obvious way to higher dimensions by writing  $\mathbb{R}^n = \mathbb{R}^3 \times \mathbb{R}^{n-3}$ , and letting  $R(t)$  be the identity on the  $\mathbb{R}^{n-3}$  factor.

A similar calculation applies to the map

$$\mathbb{R}^3 \ni \vec{x} \mapsto R\left(\alpha \ln \sqrt{x^2 + y^2 + 1}\right) \vec{x} \in \mathbb{R}^3, \quad (4.9)$$

leading again to the estimate (4.8) for  $|\alpha| < 1$ .

On  $\mathbb{R}^{2n}$ , further non-trivial maps satisfying (4.1) can be obtained by making independent logarithmic rotations (4.9) on each  $\mathbb{R}^2$  factor, etc.

## 5 Weighted Poincaré inequalities

We will need the following [7, Proposition C.2]:

PROPOSITION 5.1 *Let  $u$  be a  $C^1$  compactly supported tensor field on a Riemannian manifold  $(M, g)$ , and let  $w, v$  be two  $C^2$  functions defined on a neighborhood of the support of  $u$ . Then*

$$\int_M e^{2v} |\nabla u|^2 \geq \int_M e^{2v} [\Delta v + \Delta w + |\nabla v|^2 - |\nabla w|^2] |u|^2. \quad (5.1)$$

Throughout this section we assume that the metric  $g$  approaches the Euclidean metric as one recedes to infinity, with no decay rate imposed. The first derivatives are required to fall-off as  $o(1/r)$ . Wherever second derivatives of the metric arise in a calculation, they are required to fall-off as  $o(1/r^2)$ , etc.

We let  $r$  be any smooth positive function on  $\Omega$  bounded away from zero which coincides with the coordinate radius on  $\Omega \setminus B(R_0)$ .

Our first goal is to prove Proposition 5.4 below. As a first step, we prove:

LEMMA 5.2 *Let  $\Omega$  be as above and let  $\sigma, \mu \in \mathbb{R}$  be such that  $\sigma \neq -1/2$ . There exist constants  $c, C > 0$  such that for all  $C^2$  tensor fields  $u$  compactly supported in*

$$\Omega' := \Omega \cap \{0 < x < cr\}$$

*it holds that*

$$\int_{\Omega} x^{2\sigma+2} r^{2\mu} |\nabla u|^2 \geq C \int_{\Omega} x^{2\sigma} r^{2\mu} |u|^2. \quad (5.2)$$

REMARK 5.3 Once the inequality has been established for a metric  $g$  as explained above, it is immediate that (5.2) also holds, when  $u$  is a function, for any metric which is uniformly equivalent to  $g$ . But then it holds for tensors for all differentiable metrics uniformly equivalent to  $g$  (no decay conditions on the derivatives needed) by a standard argument, cf. e.g. [9, Remark 4.7].  $\square$

PROOF: We use (5.1) with

$$v = \sigma \ln x + \mu \ln r. \quad (5.3)$$

Then

$$\Delta v + |\nabla v|^2 = \frac{\sigma}{x^2} \left[ (\sigma-1) |\nabla x|^2 + 2\mu x \frac{\nabla x \cdot \nabla r}{r} + x \Delta x \right] + \frac{\mu}{r^2} \left[ (\mu-1) |\nabla r|^2 + r \Delta r \right]. \quad (5.4)$$

Letting

$$w = -\frac{1}{2} \ln x + \frac{\nu}{2} \ln r$$

one similarly finds

$$\Delta w - |\nabla w|^2 = \frac{1}{4x^2} \left[ |\nabla x|^2 + 2\nu x \frac{\nabla x \cdot \nabla r}{r} - 2x \Delta x \right] + \frac{\nu}{4r^2} \left[ -(\nu+2) |\nabla r|^2 + 2r \Delta r \right]. \quad (5.5)$$

Given  $\sigma \neq 1$ ,  $\mu, \nu \in \mathbb{R}$ ,  $\epsilon > 0$ , and a compact set  $K$ , for small  $x$  the dominant contribution comes from the  $|\nabla x|^2/x^2$  terms. This shows that we can choose  $x_0 > 0$  small enough so that for  $0 < x < x_0$  on  $K$  we have

$$\Delta v + |\nabla v|^2 + \Delta w - |\nabla w|^2 \geq \frac{(2\sigma-1)^2 - \epsilon}{4x^2} |\nabla x|^2. \quad (5.6)$$

We consider now  $p \in \Omega$  with

$$\lambda := r(p) \geq 2R_0, \quad (5.7)$$

Then the scaling transformation  $\mathbb{R}^n \ni \vec{y} \mapsto 2R_0\lambda^{-1}\vec{y}$  maps the set

$$\Gamma_\lambda := \Omega \cap \{\lambda/2 \leq |\vec{y}| \leq 2\lambda\}$$

to  $\Gamma_1 = \Omega \cap \{R_0 \leq |\vec{y}| \leq 4R_0\}$ . The associated pull-back will carry the metric  $g$  on  $\Gamma_{\theta,\lambda}$  to a metric  $\lambda^{-2}g_\lambda$  on  $\Gamma_{\theta,1}$ , with  $g_\lambda$  approaching the flat metric on  $\Gamma_1$  as  $\lambda$  tends to infinity. We can use (5.6) for all rescaled metrics  $g_\lambda$ , with perhaps a smaller constant  $x_0$  independent of  $\lambda$ . Since both sides are invariant under the transformation  $(x, r) \mapsto (ax, ar)$ , scaling back to  $\Omega_\lambda$  gives there

$$\Delta v + |\nabla v|^2 + \Delta w - |\nabla w|^2 \geq \frac{(\sigma - \frac{1}{2})^2}{2x^2}, \quad (5.8)$$

for all  $0 < x/r < c$  small enough, as desired.  $\square$

We are ready to prove now:

**PROPOSITION 5.4** *Let  $\Omega$  be as above and let  $\sigma, \mu \in \mathbb{R}$  be such that  $\sigma \neq -1/2$  and  $\sigma + \mu + n/2 \neq 0$ . There exist constants  $x_0, C, R$  such that for all  $C^2$  tensor fields  $u$  compactly supported in  $\Omega$  it holds that*

$$\begin{aligned} \int_{\Omega} x^{2\sigma} r^{2\mu} |u|^2 &\leq C \left( \int_{\{x \geq x_0, r=R\}} |u|^2 + \int_{\{x \geq x_0, r \leq R\}} |u|^2 \right. \\ &\quad \left. + \int_{\Omega} x^{2\sigma+2} r^{2\mu} |\nabla u|^2 \right). \end{aligned} \quad (5.9)$$

**PROOF:** We will use a family of cut-off functions  $\Xi_\lambda$  satisfying

$$|\phi \nabla \Xi_\lambda| \leq C/\lambda, \quad (5.10)$$

for  $\lambda$  large. Indeed, let  $\chi_\lambda(t) = \chi(-\ln(t)/\lambda)$  with  $\chi$  a smooth non-decreasing function satisfying  $\chi(s) = 0$  for  $s < 1$  and  $\chi(s) = 1$  for  $s \geq 2$ . Recall that in the current case  $\phi = x$ . We set  $\Xi_\lambda = \chi_\lambda(x/r)$ , then

$$x \nabla \Xi_\lambda = x \nabla (\chi(-\ln(x/r)/\lambda)) = \frac{\chi'}{\lambda} \left( \frac{x}{r} \nabla r - \nabla x \right),$$

which is bounded on  $\Omega$ . Hence (5.10) holds.

We note that  $\Xi_\lambda = 0$  if and only if  $x \geq e^{-\lambda}r$ , and  $\Xi_\lambda = 1$  if and only if  $x \leq e^{-2\lambda}r$ . Set

$$u = \Xi_\lambda u + (1 - \Xi_\lambda)u =: v + w,$$

thus  $v$  vanishes outside  $\{0 < x \leq re^{-\lambda}\}$ , and  $w$  vanishes for  $x \leq e^{-2\lambda}r$ .

Let

$$\psi = x^\sigma r^\mu.$$

Applying Lemma 5.2 to  $v$  we obtain, for all  $\lambda$  large enough,

$$\int_{\{0 < x \leq re^{-\lambda}\}} \psi^2 \phi^2 |\nabla u|^2 + \frac{c_1}{\lambda^2} \int_{\{0 < x \leq re^{-\lambda}, \nabla \Xi_\lambda \neq 0\}} \psi^2 |u|^2 \geq c_2 \int_{\{v \neq 0\}} \psi^2 |v|^2. \quad (5.11)$$

On each half-ray we have the classical Poincaré inequality for tensor fields with compact support in  $[R, \infty) \subset \mathbb{R}$ : if  $2\mu + 2\sigma + n \neq 0$ , then

$$\int_R^\infty r^{2\mu+2\sigma+n-1} |u|^2 dr \leq C_1 |u|^2(R) + C_2 \int_R^\infty r^{2\mu+2\sigma+n+1} |\partial_r u|^2 dr, \quad (5.12)$$

with constants  $C_2$  depending only upon  $2\mu + 2\sigma + n$ , and  $C_1$  depending only upon  $2\mu + 2\sigma + n$  and  $R$ . (In fact,  $C_1 < 0$  when  $2\mu + 2\sigma + n > 0$ , but this will not be needed in our considerations.)

Integrating over the angles with  $u$  replaced by  $w$ , this gives

$$\int_{\{w \neq 0\}} r^{2\mu+2\sigma+2} |\nabla w|^2 + \int_{\{w \neq 0\} \cap \{r=R\}} |w|^2 \geq C \int_{\{w \neq 0\} \cap \{r \geq R\}} r^{2\sigma+2\mu} |w|^2. \quad (5.13)$$

This implies

$$\begin{aligned} & \int_{\{w \neq 0\}} r^{2\mu+2\sigma+2} |\nabla u|^2 + \frac{c_3}{\lambda^2} \int_{\{w \neq 0, \nabla \Xi_\lambda \neq 0\}} r^{2\mu+2\sigma} |u|^2 \\ & + \int_{\{w \neq 0\} \cap \{r=R\}} |u|^2 + \int_{\{w \neq 0\} \cap \{r \leq R\}} |u|^2 \geq c_4 \int_{\{w \neq 0\}} r^{2\sigma+2\mu} |w|^2. \end{aligned} \quad (5.14)$$

On the support of  $w$  and  $\nabla \Xi_\lambda$  we have  $e^{-2\lambda}r \leq x \leq e^{-\lambda}r$ , hence

$$\begin{aligned} & \frac{1}{2} \int x^{2\sigma} r^{2\mu} |u|^2 \leq \int x^{2\sigma} r^{2\mu} |v|^2 + \int x^{2\sigma} r^{2\mu} |w|^2 \\ & \leq \int x^{2\sigma} r^{2\mu} |v|^2 + \max(e^{\lambda\sigma}, e^{2\lambda\sigma}) \int r^{2\sigma+2\mu} |w|^2 \\ & \leq \frac{C_3}{2} \left[ \int_{\{0 < x \leq re^{-\lambda}\}} x^{2\sigma+2} r^{2\mu} |\nabla u|^2 + \frac{c_1}{\lambda^2} \int_{\{0 < x \leq re^{-\lambda}\}} x^{2\sigma+2} r^{2\mu} |u|^2 \right. \\ & \quad + \max(e^{\lambda\sigma}, e^{2\lambda\sigma}) \left( \int_{\{w \neq 0\} \cap \{r=R\}} |u|^2 + \int_{\{w \neq 0\} \cap \{r \leq R\}} |u|^2 \right. \\ & \quad \left. \left. + \int_{\{x \geq re^{-2\lambda}\}} r^{2\mu+2\sigma+2} |\nabla u|^2 + \frac{c_3}{\lambda^2} \int_{\{re^{-2\lambda} \leq x \leq re^{-\lambda}\}} r^{2\mu+2\sigma} |u|^2 \right) \right]. \end{aligned}$$

For  $\lambda \geq 1$  we can rewrite this as

$$\begin{aligned} & \int x^{2\sigma} r^{2\mu} \left( 1 - C_3 \frac{c_1}{\lambda^2} \right) |u|^2 \\ & \leq C_3 \left[ c_3 \max(e^{\lambda\sigma}, e^{2\lambda\sigma}) \left( \int_{\{re^{-2\lambda} \leq x \leq re^{-\lambda}\}} r^{2\mu+2\sigma} |u|^2 + \int_{\{w \neq 0\} \cap \{r=R\}} |u|^2 \right. \right. \\ & \quad \left. \left. + \int_{\{w \neq 0\} \cap \{r \leq R\}} |u|^2 + \int_{\{x \geq re^{-2\lambda}\}} r^{2\mu+2\sigma+2} |\nabla u|^2 \right) \right. \\ & \quad \left. + \int_{\{0 < x \leq re^{-\lambda}\}} x^{2\sigma+2} r^{2\mu} |\nabla u|^2 \right]. \end{aligned}$$

Choosing  $\lambda$  large enough we obtain, keeping in mind that  $x < Cr$  on  $\Omega$ ,

$$\begin{aligned} \int x^{2\sigma} r^{2\mu} |u|^2 &\leq C_4(\lambda) \left( \int_{\{re^{-2\lambda} \leq x \leq re^{-\lambda}\}} r^{2\mu+2\sigma} |u|^2 + \int_{\{w \neq 0\} \cap \{r=R\}} |u|^2 \right. \\ &\quad \left. + \int_{\{w \neq 0\} \cap \{r \leq R\}} |u|^2 + \int_M x^{2\sigma+2} r^{2\mu} |\nabla u|^2 \right). \end{aligned} \quad (5.15)$$

Integrating (5.12) over the angles gives

$$\begin{aligned} &\int_{\{re^{-2\lambda} \leq x \leq re^{-\lambda}, r \geq R\}} r^{2\mu+2\sigma} |u|^2 \\ &\leq C(R) \int_{\{re^{-2\lambda} \leq x \leq re^{-\lambda}, r \geq R\}} r^{2\mu+2\sigma+2} |\nabla u|^2 + C_1(R) \int_{\{re^{-2\lambda} \leq x \leq re^{-\lambda}, r=R\}} |u|^2 \\ &\leq C'(R, \lambda) \int_{\{re^{-2\lambda} \leq x \leq re^{-\lambda}\}} x^{2\sigma+2} r^{2\mu} |\nabla u|^2 + C_1(R) \int_{\{re^{-2\lambda} \leq x \leq re^{-\lambda}, r=R\}} |u|^2 \\ &\leq C'(R, \lambda) \int_M x^{2\sigma+2} r^{2\mu} |\nabla u|^2 + C_1(R) \int_{\{re^{-2\lambda} \leq x \leq re^{-\lambda}, r=R\}} |u|^2. \end{aligned} \quad (5.16)$$

Inserting this into the first line of (5.15) gives (5.9).  $\square$

We note a standard consequence of Proposition 5.4:

**COROLLARY 5.5** *Let  $\Omega$  be as above and let  $\sigma, \mu \in \mathbb{R}$  be such that  $\sigma \neq -1/2$  and  $\sigma + \mu + n/2 \neq 0$ . Let  $\dot{H}$  denote the completion of  $C^2$  compactly supported tensor fields in  $\Omega$  with respect to the norm*

$$\|u\|_{\dot{H}}^2 = \int_{\Omega} x^{2\sigma} r^{2\mu} |u|^2 + \int_{\Omega} x^{2\sigma+2} r^{2\mu} |\nabla u|^2.$$

*Suppose that  $\dot{H}$  contains a closed subspace  $X$  transversal to the space  $\{u \in \dot{H} : \nabla u = 0\}$ . Then there exists a constant  $C$  such that for all  $u \in X$  we have*

$$\int_{\Omega} x^{2\sigma} r^{2\mu} |u|^2 \leq C \int_{\Omega} x^{2\sigma+2} r^{2\mu} |\nabla u|^2. \quad (5.17)$$

**PROOF:** As already mentioned, the result is standard, we give the proof for completeness.

Suppose that (5.17) is wrong, then there exists a sequence of  $C^2$  compactly supported tensor fields  $u_n$  such that

$$\int_{\Omega} x^{2\sigma} r^{2\mu} |u_n|^2 \geq n \int_{\Omega} x^{2\sigma+2} r^{2\mu} |\nabla u_n|^2. \quad (5.18)$$

We can normalize the sequence  $u_n$  so that

$$\int_{\{x \geq x_0, r=R\}} |u_n|^2 + \int_{\{x \geq x_0, r \leq R\}} |u_n|^2 = 1. \quad (5.19)$$

Equation (5.9) implies

$$\int_{\Omega} x^{2\sigma} r^{2\mu} |u_n|^2 \leq C \left( 1 + \frac{1}{n} \int_{\Omega} x^{2\sigma} r^{2\mu} |u_n|^2 \right).$$

Thus, for  $n \geq 2C$ ,

$$\int_{\Omega} x^{2\sigma} r^{2\mu} |u_n|^2 \leq 2C. \quad (5.20)$$

Equation (5.18) gives now

$$\int_{\{x \geq x_0, r \leq R\}} |\nabla u_n|^2 \leq C_1 \int_{\Omega} x^{2\sigma+2} r^{2\mu} |\nabla u_n|^2 \leq \frac{2CC_1}{n}. \quad (5.21)$$

Let  $K = \Omega \cap \{x > x_0, r < R\}$ . Compactness of the embeddings

$$W^{1,2}(K) \subset L^2(K), \quad W^{1,2}(K) \subset L^2(\{r = R\} \cap K),$$

implies that  $\{u_n\}$  contains a subsequence, still denoted by  $\{u_n\}$ , which is Cauchy both in  $L^2(K)$  and in  $L^2(\{r = R\} \cap K)$ . Equation (5.9) applied to  $u_n - u_m$  shows that  $\{u_n\}$  is Cauchy in  $\dot{H}$ . The limit is a non-trivial tensor field satisfying  $\nabla u = 0$ , which contradicts the fact that zero is the only such tensor field in  $X$ .  $\square$

We have the exponentially-weighted version of the above:

**PROPOSITION 5.6** *Let  $\Omega$  be as above and let  $s, \sigma, \mu \in \mathbb{R}$  satisfy  $s \neq 0$  and  $\beta \equiv \sigma + \mu + n/2 \neq 0$ . There exist constants  $x_0, C, R$  such that for all  $C^2$  tensor fields  $u$  compactly supported in  $\Omega$  it holds that*

$$\begin{aligned} \int_{\Omega} x^{2\sigma} r^{2\mu} e^{-sr/x} |u|^2 &\leq C \left( \int_{\{x \geq x_0, r=R\}} |u|^2 + \int_{\{x \geq x_0, r \leq R\}} |u|^2 \right. \\ &\quad \left. + \int_{\Omega} x^{2\sigma+4} r^{2\mu-2} e^{-sr/x} |\nabla u|^2 \right). \end{aligned} \quad (5.22)$$

**PROOF:** The proof is a direct repetition of that of Proposition 5.4, with  $\psi = x^{\sigma} r^{\mu} e^{-sr/(2x)}$ ,  $\phi = x^2/r$ , and where instead of Lemma 5.2 the following Lemma is used:  $\square$

**LEMMA 5.7** *Let  $\Omega$  be as above and let  $s, \mu \in \mathbb{R}$  be such that  $s \neq 0$ . There exist constants  $c, C > 0$  such that for all  $C^2$  tensor fields  $u$  compactly supported in*

$$\Omega' := \Omega \cap \{0 < x < cr\}$$

*it holds that*

$$\int_M x^{4+2\sigma} r^{2\mu-2} e^{-sr/x} |\nabla u|^2 \geq C \int_M x^{2\sigma} r^{2\mu} e^{-sr/x} |u|^2. \quad (5.23)$$

**PROOF:** We use again (5.1), with  $w = 0$  and

$$v = -\frac{sr}{x} + (2 + \sigma) \ln x + (\mu - 1) \ln r. \quad (5.24)$$

On any compact set one finds, for  $0 < x/r$  small enough,

$$\Delta v + |\nabla v|^2 = \left( \frac{s^2 r^2}{x^4} + O(rx^{-3}) + O(x^{-2}) + O(r^{-2}) \right) |\nabla x|^2. \quad (5.25)$$

By scaling, for  $s \neq 0$  and for  $0 < x < cr$  with  $c$  small enough, this implies

$$\Delta v + |\nabla v|^2 \geq \hat{C} \frac{r^2}{x^4}, \quad (5.26)$$

for some constant  $\hat{C} > 0$ , which together with (5.1) leads to (5.23).  $\square$



## 6 Applications

In this section we wish to show how to arrange things so that all conditions of Theorem 3.1 are met.

Consider, then, two smooth metrics  $g, \hat{g} \in M_{\delta+C_{r,r^\epsilon}^{k+4}} \cap C^\infty$ ,  $i = 1, 2$ , for some  $\epsilon > 0$  and  $k > n/2$ . Let  $\Omega, \Omega_S$  be as in Section 3, cf. (3.1)-(3.2).

Let  $\psi_{\vec{y}}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  denote the translation by  $\vec{y}$ ,

$$\phi_{\vec{y}}(\vec{x}) := \vec{x} + \vec{y}.$$

When  $|\vec{y}|$  tends to infinity the three tensor fields  $\phi_{\vec{y}}^*g - \delta$ ,  $\phi_{\vec{y}}^*\hat{g} - \delta$ , and  $\phi_{\vec{y}}^*(g - \hat{g})$  approach zero in  $C_{r,r^\epsilon}^{k+4}(\Omega)$ . Thus Theorem 3.1 applies for all  $|\vec{y}|$  large enough, and provides a gluing of  $\phi_{\vec{y}}^*g$  with  $\phi_{\vec{y}}^*\hat{g}$ . Taking  $\Omega$  to be a translation of the asymptotic cone as in (3.4) proves Theorem 1.1.

An alternative construction proceeds as follows, assuming that  $\Omega \cap B(0, 1) = \emptyset$ : For  $\lambda \geq 1$  let  $\psi_\lambda: \mathbb{R}^n \rightarrow \mathbb{R}^n$  denote the scaling map,  $\psi_\lambda(\vec{x}) := \lambda\vec{x}$ . Set

$$g_\lambda = \lambda^{-2}\psi_\lambda^*g, \quad \hat{g}_\lambda = \lambda^{-2}\psi_\lambda^*\hat{g}. \quad (6.1)$$

Then both  $g_\lambda$  and  $\hat{g}_\lambda$  tend to  $\delta$  in  $C_{r,r^\epsilon}^{k+4}(\Omega)$  as  $\lambda$  tends to infinity. Hence Theorem 3.1 applies for all  $\lambda$  large enough. Scaling back the glued metric provides a gluing of  $g$  and  $\hat{g}$  along a rescaled set  $\Omega$ .

Consider, finally, a collection  $\Omega_i$ ,  $i = 1, \dots, N$ , of disjoint sets satisfying the requirements set forth at the beginning of Section 3, possibly after translations. The construction just given can be repeated  $N$ -times to obtain a gluing of  $g$  and  $\hat{g}$  across  $\cup \Omega_i$ .

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